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A Schrödinger type explanation of Fresnel formulas

R. E. NISTOR[a*], I.-IOVITZ POPESCU[b], N. IONESCU-PALLAS[b]
[a]*Physics Department, Politehnica University, Bucharest, Romania*
[b]*National Institute for Lasers, Plasmas and Radiation Physics, PO Box MG7, 76900, Magurele, Romania*

The electromagnetic phenomena of reflection and refraction at a dielectric surface will be treated by an alternative Schrödinger type quantum mechanical method. Thus, a new and simple deduction of Fresnel's formulas is given, without any appeal to Maxwell's equations, by using the concept of photon potential (7) associated to the refractive index of the medium. A Helmholtz-type equation (8) for a Plücker-Kayley hexavector $\vec{Q} = \vec{E} + i\Lambda \vec{H}$, where $\Lambda = \sqrt{\mu / \varepsilon}$ is the medium constant, describes completely the photon propagation through media and opens the way of much simplification in the computation of the photon passage through optoelectronic devices, as it will be shown in the present paper for the simple case of Fresnel relations and in a companion paper for the optical coupling [10].

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1. Introduction

It happens that physical phenomena, well understood by classical concepts and methods, can be much easier investigated by a rather different conceptual and mathematical instrument, borrowed by analogy from apparently not related phenomena. Thus, in the present paper, the electromagnetic phenomena of reflection and refraction at a dielectric surface will be treated by quantum wave mechanical methods and the corresponding Fresnel relations will be derived.

As it is well known, the description of the interaction between the electromagnetic waves and the propagation medium can be achieved in many ways, by various physical models of both waves and media. Generally, inasmuch as the applications in electronics went from low frequency electromagnetic waves to higher and higher frequencies, up to the optical domain, the related theories followed the same path. However, difficulties arrose due to the fact that the Maxwell equations, so well suited at low frequencies, become inoperative at high frequencies, such as for X-rays, to say nothing about gamma-rays. To be more specific, by increasing the wave frequency, we have to conceive the light rather as photons than as electromagnetic waves, the optical domain laying approximately at the "midway" between these dual concepts. Indeed, the well known wave - particle duality expresses exactly this fact and allows us to use at optical frequencies both wave and photon concepts. The question naturally arrises if the photon concept can be used in ackowledged domains governed by the wave concept and by Maxwell equations, such as the refraction phenomenon. As far as it is deeply fixed in the scientific literature, the understanding of the refraction and diffraction phenomena is based on the concept of wave front and on the Huygens principle and has nothing to do with photons. This fundamental difficulty has been however exceeded in the second half of the XX-th century with the help of the concept of the quantum potential of the medium relative to the photon [1÷3], as it will be shown in continuation below. Moreover, Heisenberg [4] specifies that the tunneling phenomenon of quantum mechanics is completely analogous to that well known in optics of the total reflexion of a light beam on a thin metallic foil, an important fraction of the light crossing the foil when its thickness becomes of the order of the light wavelength.

2. The potential of the medium

The propagation of a scalar wave through a homogeneous, isotrope, transparent and non-dissipative dielectric medium is described in cartesian coordinates by the equation

$$\Delta\Psi - \frac{1}{v^2}\frac{\partial^2\Psi}{\partial t^2} = 0 \tag{1}$$

where

$$\Delta = \nabla^2, \quad \nabla = \vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z};$$

$$\Psi(\vec{r},t) = \Phi(\vec{r}) \cdot F(t), \quad F(t) = F(t_0)\, e^{-\frac{i}{\hbar}E\cdot(t-t_0)}.$$

$$v = \frac{c}{n} \quad , \quad n(x,y,z) > 1 \tag{2}$$

Let us introduce in Eq.1 the quantum operators for momentum and energy

$$\nabla = \frac{i}{\hbar}\,\hat{\vec{p}} \quad , \quad \frac{\partial}{\partial t} = -\frac{i}{\hbar}\,\hat{E} \tag{3}$$

and take into consideration the fact that from Eq.2 we have for F the equation

$$-\frac{i}{\hbar}\frac{\partial F}{\partial z} = EF \tag{4}$$

From the equations (1)÷(4) it then results for $\Phi(\vec{r})$ the Helmholtz-type equation

$$\Delta\Phi(\vec{r}) + K^2\Phi(\vec{r}) = 0 \tag{5}$$

where

$$K = \frac{nE}{\hbar c} \quad , \quad E = \hbar\omega \tag{6}$$

As far as $n > 1$, we can formally define a potential of the dielectric medium, relatively to a traversing photon, by the expression

$$U = E\sqrt{n^2 - 1} \tag{7}$$

Equation (5) thus becomes

$$\Delta\Phi(\vec{r}) + \frac{\left(E^2 + U^2\right)}{\hbar^2 c^2}\Phi(\vec{r}) = 0 \tag{8}$$

For the case of a step-like potential, there appears an obvious formal analogy between the equation (8) (valid for a photon) and the Schrödinger equation (valid for an electron with charge e and rest mass m_0), the latter equation being

$$\Delta\Phi_S(\vec{r}) + K_S^2\Phi_S(\vec{r}) = 0 \tag{9}$$

where

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$$K_S = \frac{\pi}{h}(8m_0)^{1/2} \cdot \sqrt{E - eV} \tag{10}$$

The electron with electric charge e couples with the step-like potential V, leading to a variation of the wave number K_S from the value

$$K_S^{(0)} = \frac{\pi}{h}(8m_0)^{1/2} \cdot \sqrt{E} \tag{11}$$

before the potential step, to the value K_S after crossing the step discontinuity. Consequently, there is an electron probability W' of crossing the potential step and an electron probability W'' of back scattering [5], namely

$$W' = \frac{4K_S K_S^{(0)}}{\left(K_S + K_S^{(0)}\right)^2} = \frac{4\sqrt{E}\cdot\sqrt{E - eV}}{\left(\sqrt{E} + \sqrt{E - eV}\right)^2}$$

$$W'' = \frac{\left(K_S - K_S^{(0)}\right)^2}{\left(K_S + K_S^{(0)}\right)^2} = \frac{\left(\sqrt{E - eV} - \sqrt{E}\right)^2}{\left(\sqrt{E - eV} + \sqrt{E}\right)^2} \tag{12}$$

$$W' + W'' = 1$$

The close formal similarity of the Helmholtz type equations (5) and (9) explains the passage of a photon from vacuum into a transparent dielectric by a simple analogy with the passage of the electron through the potential step. The lack of electric charge and of rest mass of the photon is compensated here by the refractive index of the dielectric medium, with which the photon couples via Eq. (8). Consequently, there is a photon probability W' of transmission into the dielectric and a photon probability W'' of reflecting back at the vacuum - dielectric surface, namely

$$\begin{cases} W' = \dfrac{4KK^{(0)}}{\left(K + K^{(0)}\right)^2} = \dfrac{4n}{(n+1)^2} \qquad \begin{cases} K = K^{(0)}\cdot n \;,\; K^{(0)} = \dfrac{\omega}{c} \\ W' + W'' = 1 \end{cases} \\ W'' = \dfrac{\left(K - K^{(0)}\right)^2}{\left(K + K^{(0)}\right)^2} = \dfrac{(n-1)^2}{(n+1)^2} \end{cases} \tag{13}$$

Let us now turn back to equation (8) and consider the passage of a photon with momentum $\vec{p}$. Thus, we get

$$\Delta\Phi + \frac{\vec{p}^{\,2}}{\hbar^2}\Phi = 0 \tag{14}$$

Eliminating the term $\Delta\Phi$ between Eqs. (8) and (14) we obtain the equation of propagation of the photon through the dielectric

$$\vec{p}^{\,2} = \frac{1}{c^2}\left(E^2 + U^2\right) \tag{15}$$

that is

.

$$p = n\frac{E}{c} \quad ; \quad E = \frac{c}{n}p = v \cdot p \tag{16}$$

We conclude, therefore, that assumption (7) and equation (8) are correct and eventually lead to the probabilities W' and W'' given by Eq.13 in a similar way with that of the passage of an electron through a potential barrier.

3. Dielectric interfaces

When a particle crosses the interface between two media with potentials U_1 and U_2 respectively, the parallel momentum component to this interface is conserved [5], that is

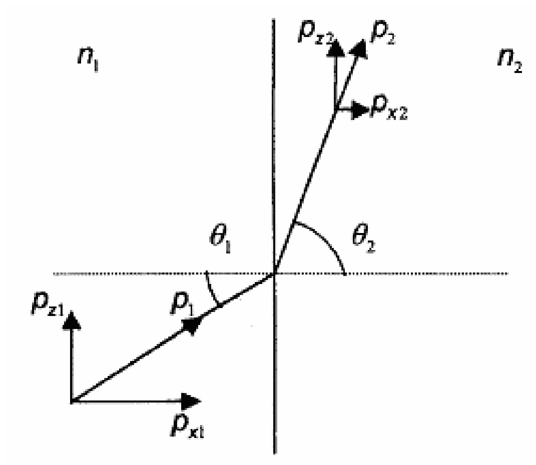

Fig. 1. The photon momenta in two dielectric media with $n_2 < n_1$.

$$\vec{p}_{z1} = \vec{p}_{z2} \tag{17}$$

Taking further into account that $\vec{p} = \vec{p}_0 n$, we can write in terms of the angles of Fig. 1

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \tag{18}$$

hence the light refraction law. On the other hand, the normal momentum component to the interface between the two media is not conserved and is given by

$$p_{x1,2} = p_{1,2} \cos\theta_{1,2} \tag{19}$$

or, in terms of the photon momentum in vacuum,

$$p_{x1,2} = p_0 n_{1,2} \cos\theta_{1,2} \tag{20}$$

Using further Eq.15, we can say that the photon “sees” in the $\overrightarrow{Ox}$ direction the potentials

$$U_{x1,2}^2 = p_{x1,2}^2 c^2 - E^2 \tag{21}$$

Taking into account Eq.20, the last expression becomes

$$U_{x1,2}^2 = E^2\left(n_{1,2}^2 \cos^2\theta_{1,2} - 1\right) \tag{22}$$

or, with the help of the refraction law (18), we can express the potentials in terms of the sinus of the incidence angle, that is

$$U_{x1,2} = E\sqrt{n_{1,2}^2 - n_1^2 \sin^2\theta_1 - 1} \tag{23}$$

4. The photons and their wave function

Generally, the term of photon coordinate has no physical meaning and, consequently, the concept of photon wave function has not the meaning of an amplitude of spatial localization as meant in the non-relativistic quantum mechanics. However, the photon momentum is a measurable quantity and the photon wave function in the momentum representation has a deep meaning and allows computing the photon momentum and polarization probabilities [6]. Thus, let us consider in Fig. 2 the step potential in the

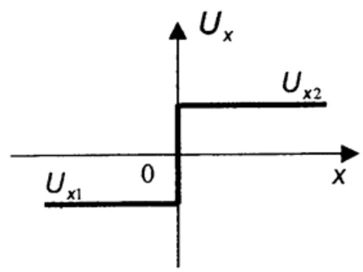

Fig. 2. The step potential in the $\overrightarrow{Ox}$ direction, normal to the dielectrics interface.

direction $\overrightarrow{Ox}$ normal to the dielectrics interface. The corresponding wave functions in the momentum representation, solutions of Eq.8, are

$$x \le 0\,; \quad \Phi_1 = A e^{\frac{i}{\hbar} p_{x1} x} + B e^{-\frac{i}{\hbar} p_{x1} x} \tag{24}$$

$$x \ge 0\,; \quad \Phi_2 = C e^{\frac{i}{\hbar} p_{x2} x} \tag{25}$$

Here we assumed that the incident photon arrives at the interface from the left side of Fig. 2.

5. Comments and conclusions

The main purpose of the present paper consists in finding out a further theoretical motivation for the concept of “photon tunneling”. Though it has been already used in some recent applications [7,8] and has been referenced in a book on quantum mechanics [9] and, indirectly, even by Heisenberg [4], the "photon tunneling" concept still misses a clear theoretical basis. This complex interaction of the light with dielectric media has found a completely satisfactory explanation in the framework of the classical electrodynamics, yet it has been reluctant so far to a formal quantum mechanical treatment, inasmuch as the photon has neither electric charge nor rest mass. Consequently, the photon cannot "feel" a potential step as it is the case for the electron. Fortunately, however, as it has been demonstrated in the present paper, the photon "feels" the variations of the refraction index of the medium as variations of the wave number, in a completely similar way as the electron does for the potential variations. This is the theoretical principle which converts the "photon tunnelling" from a simple metaphor (as currently believed) into a completely operational opto-quantum analogy.

A simple application o this analogy will be the deduction of Fresnel formulas starting from the Helmholtz-type scalar tridimensional equation derived above

$$(\Delta + K_0^2 n^2)\Phi(x, y, z) = 0 \tag{26}$$

where, however, the scalar function Φ has to be replaced by the hexavector $\vec{Q} = \vec{E} + i\Lambda \vec{H}$ (where $\Lambda = \sqrt{\mu / \varepsilon}$ is the medium constant) in order to ensure the transverse propagation of the electromagnetic waves. The details of the computation of reflection and transmission amplitudes for various polarizations are given in the Appendix. The advantages of the Schrödinger type treatment of the photon passage through optical media will be further demonstrated in the companion paper, devoted to optical coupling [10].

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*Corresponding author: Rudolf Nistor nistor@physics.pub.ro

Appendix

Let us denote the operator $\Delta + K_0^2 n^2$ by

$$\hat{O} = \Delta + K_0^2 n^2 \tag{A1}$$

and write Eq. (26) as

$$\hat{O}\,\Phi = 0 \tag{A2}$$

On the other hand, from the Plücker-Kayley geometry of a free vectorial field [11], the associated hexavector

$$\vec{Q} = \mathrm{Re}\,\vec{Q} + i\,\mathrm{Im}\,\vec{Q} \tag{A3}$$

has the property

$$\vec{Q}^{\,2} = 0 \tag{A4}$$

By identification, from Eq (A1) and Eq. (A2) we get

$$\mathrm{Re}\,\vec{Q} \cdot \mathrm{Im}\,\vec{Q} = 0 \tag{A5}$$

and

$$(\mathrm{Re}\,\vec{Q})^2 - (\mathrm{Im}\,\vec{Q})^2 = 0 \tag{A6}$$

We obtained in this way two invariants, a scalar one (A5) and a pseudoscalar the other one (A6).

Let us further assume that in an homogenous, isotropic and transparent dielectric medium there are also two invariants of the electromagnetic field, namely

$$\vec{E} \cdot \vec{H} = 0 \tag{A7}$$

$$\vec{E}^{\,2} - \Lambda^2 \vec{H}^{\,2} = 0 \tag{A8}$$

where $\Lambda = \sqrt{\mu / \varepsilon}$ is the medium constant. These latter invariants can be framed within the Plücker-Kayley geometry by setting

$$\mathrm{Re}\,\vec{Q} \equiv \vec{E} \tag{A9}$$

and

$$\mathrm{Im}\,\vec{Q} \equiv \Lambda\,\vec{H} \tag{A10}$$

The electromagnetic field in the considered medium is characterized by

- a polar vector $\vec{E}$ with orto properties at reflexion ($\overset{o}{\vec{E}} \equiv -\vec{E}$),
- an axial vector $\vec{H}$ with pseudo properties (pseudovector) at reflection ($\overset{o}{\vec{H}} \equiv \vec{H}$) and
- a propagation vector $\vec{P}$ with the property $\vec{P} \approx \vec{E} \times \vec{H}$

so that $\vec{E}$, $\vec{H}$ and $\vec{P}$ make up an ortho-grade trihedral angle. To these one should add the properties resulting from Maxwell equations, namely that the Helmholtz operator $\hat{O}$, Eq. A1, applied to $\vec{E}$ and $\vec{H}$ wanishes, that is

$$\hat{O}\,\vec{E} = 0 \tag{A11}$$

$$\hat{O}\,\vec{H} = 0 \tag{A12}$$

When the light ray passes from a medium with the refractive index n_1 in a medium with the refractive index n_2, this operator exhibits a discontinuity

$$\hat{O} = \Delta + K_0^2 n_1^2 \rightarrow \hat{O} = \Delta + K_0^2 n_2^2 \tag{A13}$$

This discontinuity should not affect the energy and momentum balance and this requires certain continuity conditions for the electric $\vec{E}$ and magnetic $\vec{H}$ fileds.

The system of the three equations, namely $\hat{O}\,\Phi = 0$, $\hat{O}\,\vec{E} = 0$ and $\hat{O}\,\vec{H} = 0$ represents the essence of the analogy between quantum mechanics and optics.

The Fresnel formulas result in a natural way from the continuity conditions of the tangential components of the vectors $\vec{E}$ and $\vec{H}$ [12], namely

$$B_{\perp} = -\frac{\sin(\theta_1 - \theta_2)}{\sin(\theta_1 + \theta_2)} A_{\perp} \tag{A14}$$

$$C_{\perp} = \frac{2\cos\theta_1 \sin\theta_2}{\sin(\theta_1 + \theta_2)} A_{\perp} \tag{A15}$$

for the normal polarization to the incidency plane and, respectively,

$$B_{\|} = A_{\|} \frac{\mathrm{tg}(\theta_1 - \theta_2)}{\mathrm{tg}(\theta_1 + \theta_2)} \tag{A16}$$

$$C_{\|} = A_{\|} \frac{2\cos\theta_1 \sin\theta_2}{\sin(\theta_1 + \theta_2)\cos(\theta_1 - \theta_2)} \tag{A17}$$

for the parallel polarization to the incidency plane, where A, B, and C are the corresponding amplitudes of the incident, reflected, and transmitted waves.